Exam Advanced Quantum Mechanics

Thursday January 22, 2015, 9:00-12:00, Room V5161.0165

Before you start, read the following:

- There are 4 problems for a total of 40 points.
- Start each problem on a new sheet of paper.
- Write your name and student number on each sheet of paper.
- Illegible handwriting will not be graded.
- *Good luck!*

Problem 1 (*45 minutes; 10 points in total*)

Consider the time-independent Schrödinger equation

$$H(\lambda)\psi(\lambda) = E(\lambda)\psi(\lambda) \ ,$$

where λ is a parameter and $E(\lambda)$ the energy eigenvalue of the Hamiltonian $H(\lambda)$.

2 pnts (a) Prove that the expectation value of the operator $\partial H/\partial\lambda$ is given by

$$\langle\frac{\partial H}{\partial\lambda}\rangle = \frac{\partial E}{\partial\lambda} \ .$$

Apply this to the radial Schrödinger equation for hydrogen-like systems, with Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} - Z\frac{a_0}{r}\alpha^2 mc^2 \ , \quad \text{where} \ \ p_r = -i\hbar\frac{1}{r}\frac{\partial}{\partial r}r \ .$$

The energy values are $E_n = -\frac{1}{2}\left(Z\alpha/n\right)^2 mc^2$, with $n = n_r + \ell + 1$ the principal quantum number. The Bohr radius is $a_0 = \hbar/(\alpha mc)$.

2 pnts (b) Use Z as parameter and calculate the expectation value $\langle 1/r\rangle$ for the state n, ℓ.

2 pnts (c) Use ℓ as parameter and calculate the expectation value $\langle 1/r^2\rangle$ for the state n, ℓ.

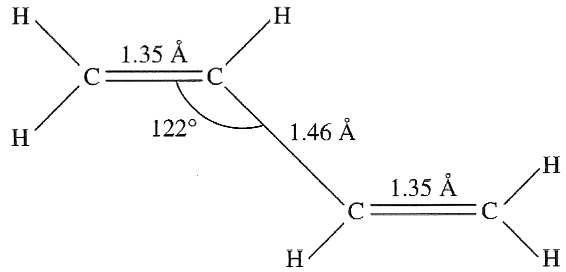
4 pnts (d) To calculate the expectation value $\langle 1/r^3\rangle$, proceed as follows:

(*i*) Evaluate the commutators $[p_r, r^n]$, $[p_r, H]$.

(*ii*) For an eigenstate of H holds that $\langle[A, H]\rangle = 0$ for any operator A. Apply this to $[p_r, H]$.

Problem 2 (*45 minutes; 10 points in total*)

The molecule C_4H_6 has a linear structure, with a skeleton $(C_4H_6)^{4+}$ of σ electrons and four carbon atoms numbered $n = 1$ to $n = 4$. A π electron localized near carbon atom n has the state vector $|\phi_n\rangle$.



We generalize to a linear chain of N carbon atoms, with labels $n = 1, \ldots, N$. The Hamiltonian of a π electron acts on the state $|\phi_n\rangle$ as

$$\begin{aligned} H|\phi_n\rangle &= E_0|\phi_n\rangle - A(|\phi_{n-1}\rangle + |\phi_{n+1}\rangle) \ , \quad \text{for } n \neq 1, N \ , \\ H|\phi_1\rangle &= E_0|\phi_1\rangle - A|\phi_2\rangle \ , \\ H|\phi_N\rangle &= E_0|\phi_N\rangle - A|\phi_{N-1}\rangle \ , \end{aligned}$$

where $A > 0$ is a constant.

3 pnts (a) Give the matrix for H in the $|\phi_n\rangle$ basis for $N = 4$. Write the most general state for a π electron as $|\chi\rangle = \sum_{n=1}^{N} c_n|\phi_n\rangle$. Give the action of H on the state $|\chi\rangle$. Introduce for this two fictitious states $|\phi_0\rangle$ and $|\phi_{N+1}\rangle$ with $c_0 = c_{N+1} = 0$ and rewrite $|\chi\rangle$ as $|\chi\rangle = \sum_{n=0}^{N+1} c_n|\phi_n\rangle$.

2 pnts (b) Find c_n in the form $c_n = c(e^{in\delta} - e^{-in\delta})/2i$, which ensures that $c_0 = 0$. Show that one must choose $\delta = \pi s/(N+1)$, with $s = 1, \ldots, N$.

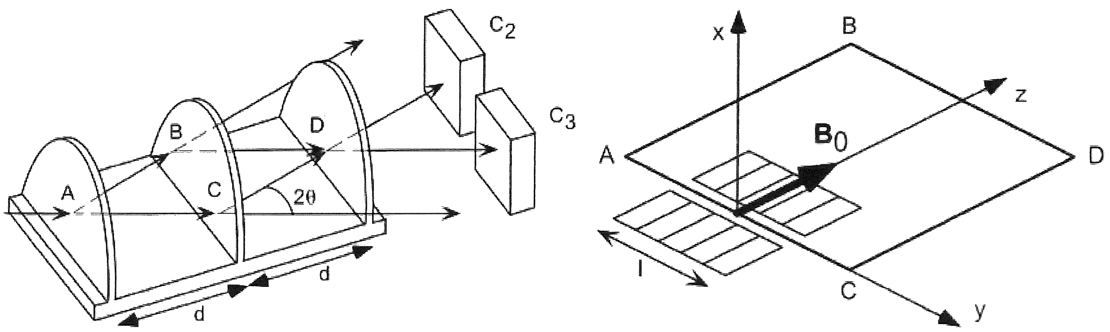
3 pnts (c) Show that the eigenvalues of H are $E_s = E_0 - 2A\cos[\pi s/(N+1)]$. For $N = 4$, show that the ground-state energy of the four π electrons is $E \simeq 4(E_0 - A) - 0.48A$. Discuss the effect of delocalization.

2 pnts (d) Show that the normalization constant in (b) is $c = \sqrt{2/(N+1)}$.

$\cos(\pi/5) \simeq 0.81$
$\cos(2\pi/5) \simeq 0.31$

Problem 3 (*45 minutes; 10 points in total*)

In the neutron interferometer below the neutron beam is monochromatic with de Broglie wavelength λ. The silicon strips split the incident plane waves into two waves with equal probability. The amplitudes for the two paths ACD and ABD interfere at D. The neutrons are detected at the counters C_2 and C_3.



In the arm AC a constant magnetic field $\vec{B} = B\hat{z}$ acts in the plane ABCD over a length l. Its strength B can be varied.

2 pnts (a) Show that the time that it takes the neutron to traverse the distance l is given by $T = Ml\lambda/(2\pi\hbar)$, where M is the neutron mass.

When the neutron goes from a point 1 to another point 2, the amplitudes are related by the time-evolution equation $\psi_2 = e^{-iHt/\hbar}\psi_1$. The Hamiltonian in the presence of the magnetic field is $H = p^2/(2M) - \mu_n \vec{\sigma} \cdot \vec{B}$.

3 pnts (b) Calculate the phase $e^{-i\varphi} = e^{-iHT/\hbar}$ that the amplitude for the path ACD picks up due to the spin precession in the magnetic field.

Assume that when $B = 0$ the counting rates at C_2 and C_3 are equal.

3 pnts (c) Prove that the difference in the magnetic fields that produce two successive maxima in the counting rates is given by $\Delta B = 4\pi^2\hbar^2/(|\mu_n|Ml\lambda)$.

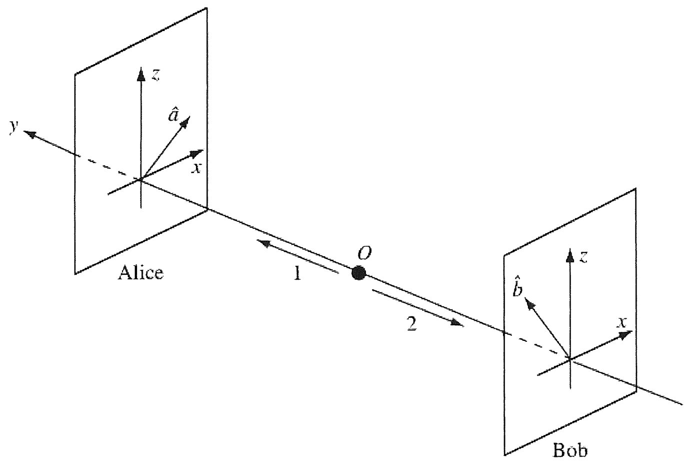
2 pnts (d) Explain how this setup can be used to prove that the state vector of a spin-1/2 neutron changes sign under a rotation by an odd multiple of 2π.

Problem 4 (*45 minutes; 10 points in total*)

A source emits pairs of spin-1/2 particles in opposite directions. Alice and Bob measure the spin components of particle 1 and 2 along axes $\hat{a}$ and $\hat{b}$ in the $\hat{x}$-$\hat{z}$ plane. The angle between $\hat{a}$ and $\hat{b}$ is θ. The spin state of the pairs is

$$|\Phi\rangle = \left(|+_1 \otimes -_2\rangle - |-_1 \otimes +_2\rangle\right)/\sqrt{2}\,,$$

where $|\pm\rangle$ are the eigenstates of S_z.



2 pnts (a) Explain what an entangled state is. Show that $|\Phi\rangle$ is an entangled state for particles 1 and 2. Show that $|\Phi\rangle$ is invariant under rotations.

2 pnts (b) Take $\hat{a} = \hat{b} = \hat{z}$ and evaluate $[(S_1)_z \otimes (S_2)_z]\,|\Phi\rangle$. What do Alice and Bob conclude when comparing their results? What is $[(S_1)_x \otimes (S_2)_x]\,|\Phi\rangle$?

Denote the possible results of the measurements of $\vec{\sigma}_1 \cdot \hat{a}$ and $\vec{\sigma}_2 \cdot \hat{b}$ by $\varepsilon_a = \pm 1$ and $\varepsilon_b = \pm 1$, respectively. The correlation is defined by

$$E(\hat{a}, \hat{b}) = \langle \varepsilon_a \varepsilon_b \rangle = \sum_{\varepsilon_a, \varepsilon_b} \varepsilon_a \varepsilon_b P_{\varepsilon_a \varepsilon_b}\,,$$

where $P_{\varepsilon_a \varepsilon_b}$ is the joint probability for Alice to find ε_a and Bob to find ε_b.

3 pnts (c) Argue that one can assume that $\hat{a} = \hat{z}$. The eigenstates with $\varepsilon_b = \pm 1$ are denoted $|\pm, \hat{b}\rangle$. Show that $|+\otimes[+, \hat{b}]\rangle = \cos(\theta/2)|+\otimes+\rangle + \sin(\theta/2)|+\otimes-\rangle$. Calculate the amplitude $a_{++} = \langle +\otimes\,[+, \hat{b}]|\Phi\rangle$.

3 pnts (d) Calculate P_{++}, P_{+-}, P_{-+}, and P_{--} as function of θ. Prove that $E(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}$.

$$D^{(1/2)}(\theta, \phi) = \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) & -e^{-i\phi/2}\sin(\theta/2) \\ e^{i\phi/2}\sin(\theta/2) & e^{i\phi/2}\cos(\theta/2) \end{pmatrix}$$